PHYSICAL REVIEW E VOLUME 50, NUMBER 6 DECEMBER 1994

Analysis of resonant structures of four-dimensional symplectic mappings, using normal forms

E. Todesco
Dipartimento di Fisica della Università di Bologna, Bologna, Italy
and Istituto Nazionale di Fisica Nucleare, Sezione di Bologna, Via Irnerio 46, 40126, Bologna, Italy

(Received 19 April 1994)

The geometry of the resonant orbits of symplectic four-dimensional mappings, in the neighborhood of an elliptic fixed point, is analyzed in the framework of a perturbative approach based on resonant normal forms. The analysis of the truncated interpolating Hamiltonian allows one to determine the classification, the location, and the stability of the integrable resonant structures in phase space.

PACS number(s): 03.20.+i, 46.10.+z, 95.10.−a

I. INTRODUCTION

Four-dimensional (4D) symplectic mappings are a model of a wide class of problems in different fields of physics, such as celestial mechanics and beam dynamics. Contrary to the 2D case, in 4D the structures of the phase space of a nonintegrable map are not yet completely understood, even if significant analytical and numerical results have been obtained in the last two decades [1–3]. In this paper we outline a method, based on perturbative tools, which allows one to give a complete classification of the resonant orbits of a 4D symplectic map in the neighborhood of a fixed point whose eigenvalues lie on the unit circle.

Let us first consider a symplectic 2D integrable map in the form

$$z' = T(z,z^*) = e^{i\Omega(r)}z, \quad r \equiv zz^*, \quad z \in \mathbb{C}, \tag{1}$$

i.e., a twist mapping. Let $\Omega(r) = \omega + \Omega_2 r + O(r^2)$ be a real function of r with $\Omega_2 \neq 0$. If the amplitude r is such that the frequency $\Omega(r)/2\pi$ is irrational, the orbits are dense on the 1D torus, i.e., on the circle $z = \sqrt{r}e^{i\vartheta}$, $\vartheta \in [0,2\pi[$. On the other hand, if the frequency is rational, $\Omega(r)/2\pi = p/q$ (throughout the paper p,q will denote integers without common divisors), then T has an infinity of parabolic fixed points $z = \sqrt{r}e^{i\vartheta}$, $\vartheta \in [0,2\pi[$ of period q.

If one considers a nonlinear perturbation which preserves the symplectic conditions, the Kolmogorov-Arnold-Moser (KAM) theorem [4] ensures that for small perturbations there exist invariant curves with "strongly irrational" (i.e., diophantine) frequency, that are deformed circles. On the other hand, the Poincaré-Birkhoff theorem [2,4] states that among the infinite set of parabolic fixed points of period q of the map T, only $2jq$ fixed points (where j is a positive integer) survive under perturbation: one has jq hyperbolic fixed points and jq elliptic fixed points of period q.

The KAM theorem is also valid for mappings with higher dimensionality, and a generalization of the Poincaré-Birkhoff theorem, which proves the existence of fixed points of arbitrary order in the neighborhood of an elliptic fixed point, has been given in Ref. [2]. Indeed, a phenomenological approach based on numerical analysis [5] shows that, besides fixed points and 2D invariant tori, one can have 1D resonant orbits (fixed lines). In this paper we outline the perturbative approach of resonant normal forms, which allows one to give a classification of the integrable resonant orbits (fixed points and fixed lines), and to compute their stability as a function of the polynomial expansion of the 4D symplectic mapping.

Resonant normal forms provide an effective analytical framework to analyze the numerical data; moreover, they give a nontrivial result about the stability of fixed points in the case of a double resonance [see Eq. (12)] which could hardly be obtained through numerical analysis. Owing to the asymptotic character of the series, the results are not rigorous, but are in agreement with the above-cited theorems and with numerical simulations.

II. NORMAL FORMS

The normal form approach [6–8] is the natural generalization of the canonical perturbation theory for Hamiltonian flows to symplectic mappings: given a symplectic map $\mathbf{F}$ in a $2n$-dimensional phase space, having a fixed point in the origin, one looks for a nonlinear transformation $\mathbf{\Phi}$ such that $\mathbf{F}$ is transformed to a new map $\mathbf{U}$ that is "particularly simple," i.e., that has explicit invariants and symmetries. The map $\mathbf{U}$ is the normal form. The conjugating equation of the map to its normal form reads

$$\mathbf{\Phi}^{-1} \circ \mathbf{F} \circ \mathbf{\Phi}(\zeta) = \mathbf{U}(\zeta), \tag{2}$$

where ζ are the new variables in phase space, called normal coordinates. $\mathbf{U}$ is invariant under a symmetry group generated by a linear transformation Λ_α, i.e., it commutes with Λ_α: this symmetry condition defines the normal form $\mathbf{U}$ and the conjugating function $\mathbf{\Phi}$ (up to a gauge group).

The existence of a formal solution is guaranteed by theorems that state that one can build a normal form $\mathbf{U}$ with respect to the symmetry group generated by the linear part of the map Λ_ω, or subgroups of it. Analytic solutions to the functional equation (2) in open neighborhoods of an elliptic fixed point do not exist in the generic case: the series are divergent. Indeed, one can prove that the perturbative series are asymptotic, and therefore optimal truncation can provide a very accurate approximation of the dynamics of the nonlinear map: this has allowed applications to numerous problems of celestial mechanics [3] and accelerator physics [9–12].

 © 1994 The American Physical Society

In order to analyze the geometry of the orbits of the normal forms, one can build an Hamiltonian H whose orbits interpolate the orbits of $\mathbf{U}$; since $\mathbf{U}$ commutes with the symmetry group, one has

$$H(\Lambda_\alpha \mathbf{z})=H(\mathbf{z}). \tag{3}$$

The analysis of H allows one to determine perturbative expansions for a wide class of nonlinear quantities that characterize the dynamics, such as the frequencies [10,12], the location and stability of the fixed points [8,11], and the topology of resonant orbits.

III. RESONANT STRUCTURES IN 2D

In the 2D case, in the neighborhood of an elliptic fixed point, one can build normal forms defined by the symmetry groups generated by the linear matrix $\Lambda_\alpha=\mathrm{diag}(e^{i\alpha},e^{-i\alpha})$. We shall express the normal forms and the interpolating Hamiltonian in the variables (ρ,θ), related to the normal coordinates by $\zeta=\sqrt{\rho}e^{i\theta}$. One can have two different types of normal forms: nonresonant normal forms $[\alpha/(2\pi)$ irrational], which are invariant under the group of continuous rotations, and resonant normal forms $[\alpha/(2\pi)=p/q]$, which are invariant under the group of discrete rotations by an angle of $2\pi/q$. In the first case the normal form is an amplitude-dependent rotation (i.e., a twist mapping), and the interpolating Hamiltonian H is a function of the amplitude ρ; in the second case H has the form

$$h(\rho,\theta)\equiv -iH(\zeta(\rho,\theta))=\sum_{k,l} h_{k,l}\rho^{k+lq/2}\cos(lq\theta+\varphi_{k,l}). \tag{4}$$

The coefficients $h_{k,l}$ are real and positive; one distinguishes between the coefficients $h_{k,0}$, which produce the dependence of the frequency on ρ, and the other coefficients, which excite the nonlinear resonance of order q. We consider a generic mapping with $h_{1,0}\neq 0$ and a resonance of order $q\geqslant 5$. The analysis of h gives the topology of the resonant orbits: if we truncate it at the first significant resonant term [i.e., neglecting $O(\rho^{(q+1)/2})$], we obtain a pendulum Hamiltonian which has q hyperbolic and q elliptic fixed points, in agreement with the Poincaré-Birkhoff theorem. If the first order resonant coefficient $h_{0,1}$ is zero, one has to consider the higher orders: therefore it is possible to find cases where one has $2jq$ fixed points, with j positive integer.

IV. RESONANT STRUCTURES IN 4D

Classification of normal forms

In the 4D case, the symmetry groups that define the normal form in the neighborhood of a bi-elliptic fixed point are generated by the linear matrix $\Lambda_\alpha=\mathrm{diag}(e^{i\alpha_1},e^{-i\alpha_1},e^{i\alpha_2},e^{-i\alpha_2})$. Let $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$ and $\mathbf{k}$ be a 2D vector with integer entries; the different types of solution to the equation

$$(\boldsymbol{\alpha}\cdot\mathbf{k})\bmod 2\pi=0 \tag{5}$$

define four different groups of invariance, and therefore four types of normal forms. Interpolating Hamiltonians will be expressed in the coordinates $(\rho_1,\rho_2,\theta_1,\theta_2)$, related to the normal coordinates by $\zeta_1=\sqrt{\rho_1}e^{i\theta_1}$, $\zeta_2=\sqrt{\rho_2}e^{i\theta_2}$.

Nonresonant case. If $\mathbf{k}=\mathbf{0}$ is the only solution of (5), the symmetry group is the direct product of 2D continuous rotations. The normal form is an amplitude-dependent rotation and h is a power series in the amplitudes ρ_1,ρ_2, which are the independent integrals of motion. The 2D tori $[0,2\pi[\times[0,2\pi[$ are the invariant surfaces.

Single-uncoupled resonance. If the solution of (5) is $\mathbf{k}=l(q,0)$, or $\mathbf{k}=l(0,q)$ (where $l\in\mathbb{Z}$ and $q\in\mathbb{N}$), the symmetry group generated by the linear part is given by the direct product of 2D continuous rotations times 2D discrete rotations by an angle $2\pi/q$. In the case $\mathbf{k}=l(q,0)$, h has the form

$$h(\rho_1,\rho_2,\theta_1)=\sum_{k_1,k_2,l} h_{k_1,k_2,l}(\rho_1)^{k_1+lq/2}(\rho_2)^{k_2}\times\cos(lq\theta_1+\varphi_{k_1,k_2,l}). \tag{6}$$

One has two independent integrals of motion: h and ρ_2. A similar result holds for the case $\mathbf{k}=l(0,q)$.

Single-coupled resonance. If the solution of (5) is $\mathbf{k}=l(q,p)$ (where $l,p\in\mathbb{Z}$ and $q\in\mathbb{N}$), the symmetry group is a one parameter compact group. The Hamiltonian h has the form

$$h(\rho_1,\rho_2,\theta_1,\theta_2)=\sum_{k_1,k_2,l} h_{k_1,k_2,l}(\rho_1)^{k_1+lq/2}(\rho_2)^{k_2+l|p|/2}\cos[l(q\theta_1+p\theta_2)+\varphi_{k_1,k_2,l}]. \tag{7}$$

The independent integrals of motion are $p\rho_1-q\rho_2$ and h.

Double resonance. In this case the solution of (5) is $\mathbf{k}=l_1(q_1,p_1)+l_2(p_2,q_2)$ (where $l_1,l_2,p_1,p_2\in\mathbb{Z}$ and $q_2,q_2\in\mathbb{N}$). If $p_1=p_2=0$, the symmetry group is given by the direct product of discrete 2D rotations by an angle $2\pi/q_1$ times discrete 2D rotations by an angle $2\pi/q_2$, and h reads

$$h(\rho_1,\rho_2)=\sum_{k_1,k_2,l_1,l_2}(\rho_1)^{k_1+l_1q_1/2}(\rho_2)^{k_2+l_1q_2/2}[h^+_{k_1,k_2,l_1,l_2}\cos(l_1q_1\theta_1+l_2q_2\theta_2+\varphi^+_{k_1,k_2,l_1,l_2}) + h^-_{k_1,k_2,l_1,l_2}\cos(l_1q_1\theta_1-l_2q_2\theta_2+\varphi^-_{k_1,k_2,l_1,l_2})]. \tag{8}$$

A similar result holds for the other cases. When a double resonance condition is satisfied, h is the only explicit integral of motion, and therefore the resonant normal forms do not provide a complete set of prime integrals. Numerical analysis shows that when there are stable orbits in the neighborhood of the fixed point, the second integral of motion exists for most initial conditions: an approximate value can be obtained by using nonresonant normal forms truncated at order $N<\min\{(|p_1|+q_1)/2,(|p_2|+q_2)/2\}$.

Analysis of the orbits of a 4D twist mapping

We first analyze the orbits of the 4D twist mapping:

$$
\begin{aligned}
z_1' &= T_1(\mathbf{z}) = \exp[i\Omega_1(\rho_1,\rho_2)]z_1, \quad \rho_1 = z_1 z_1^* \\
z_2' &= T_2(\mathbf{z}) = \exp[i\Omega_2(\rho_1,\rho_2)]z_2, \quad \rho_2 = z_2 z_2^* .
\end{aligned}
\tag{9}
$$

Since the map is integrable, one can compute all the relevant quantities of its orbits; the iterates always lie on the 2D torus, but according to the different values of the nonlinear frequencies, one finds different topologies.

Nonresonant case. We fix (ρ_1,ρ_2) such that $\Omega_1(\rho_1,\rho_2)$ and $\Omega_2(\rho_1,\rho_2)$ satisfy a nonresonant condition: one obtains an orbit that is dense on the torus; its closure has dimension two, and it is connected.

Single-uncoupled resonance. We fix (ρ_1,ρ_2) such that $\Omega_1(\rho_1,\rho_2)$ and $\Omega_2(\rho_1,\rho_2)$ satisfy a single-uncoupled resonant condition: one obtains an orbit that is dense on the direct product of a 1D torus times q parabolic 2D fixed points. We call these structures "single-uncoupled resonance parabolic fixed lines of period q." The closure of the orbit has dimension one, and is made up of q pieces which are connected. For each initial condition chosen in $\theta_1 \in [0,2\pi/q[, \theta_2=0$, one obtains an infinity of different parabolic fixed lines of the same period, having the same geometry.

Single-coupled resonance. We fix (ρ_1,ρ_2) such that $\Omega_1(\rho_1,\rho_2)$ and $\Omega_2(\rho_1,\rho_2)$ satisfy a single-coupled resonance condition: one obtains an orbit, which lies on the 2D torus, that is dense on the 1D curve of the equation

$$
\theta_1(t)=\theta_1+tp\nu, \quad \theta_2(t)=\theta_2+tq\nu, \quad t\in[0,2\pi[. \tag{10}
$$

We call this structure the "single-coupled resonance parabolic fixed line"; the closure of the orbit is connected. For each initial condition chosen in $\theta_1 \in [0,2\pi/p[, \theta_2=0$ one obtains an infinity of different parabolic fixed lines of the same type, having the same geometry.

Double resonance. We fix (ρ_1,ρ_2) such that $\Omega_1(\rho_1,\rho_2)$ and $\Omega_2(\rho_1,\rho_2)$ satisfy the double resonance condition explicitly considered in the previous subsection: one obtains an orbit, which lies on the 2D torus, that is made up of q_1q_2 parabolic fixed points. The dimension of the orbit is zero, and is made up of q_1q_2 components trivially connected. For each initial condition chosen in $\theta_1 \in [0,2\pi/q_1[, \theta_2 \in [0,2\pi/q_2[$, one obtains an infinity of families of fixed points.

Classification of 4D resonant structures

Following the strategy outlined for the 2D case, we have analyzed the first order resonant truncation of the interpolating Hamiltonian in order to classify the resonant structures of a symplectic 4D map. We consider a 4D twist mapping $\mathbf{T}$ [see Eq. (9)] having a family of resonant orbits in the neighborhood of the origin. We restrict ourselves to the analysis of resonances with $|p|+q\geq 5$. We conjecture that a small symplectic perturbation changes the topology of these orbits according to the following cases.

Single-uncoupled resonance. Let ρ_1,ρ_2 be positive amplitudes such that the single-uncoupled resonance condition is satisfied, i.e., there exist an infinity of single-uncoupled resonance parabolic fixed lines of period q. A generic perturbation preserves only two single-uncoupled resonance fixed lines of period q: one is elliptic and one is hyperbolic. (By generic perturbation we mean a perturbation which gives rise to a Hamiltonian with $h_{0,1}\neq 0$. Nongeneric cases give rise to a situation analogous to the 2D case with $j>1$.)

Single-coupled resonance. Let ρ_1,ρ_2 be positive amplitudes such that the single-coupled resonance condition is satisfied, i.e., there exist an infinity of single-coupled resonance parabolic fixed lines. A generic symplectic perturbation (see above) preserves only two fixed lines: one is elliptic and one is hyperbolic.

Double resonance. Let ρ_1,ρ_2 be positive amplitudes such that the double resonance condition is satisfied, i.e., there exists an infinity of parabolic fixed points of period q_1q_2. A generic symplectic perturbation (see above) preserves $4q_1q_2$ fixed points, which can be split in four families: fixed points which are obtained by iteration of the map belong to the same family. We denote by $(\rho_1^{\pm},\rho_2^{\pm})$ the amplitudes of the four families of fixed points, and we define the quantities

$$
\alpha=2h_{2,0,0,0}, \quad \beta=h_{1,1,0,0}, \quad \gamma=2h_{0,2,0,0}
$$

$$
\delta=\mp(q_1)^2h_{0,0,1,0}(\rho_1^{\pm +})^{q_1/2}, \quad \delta=\mp(q_1)^2h_{0,0,1,0}(\rho_1^{\pm -})^{q_1/2}
$$

$$
\eta=(q_2)^2h_{0,0,0,1}(\rho_2^{\mp -})^{q_2/2}, \quad \eta=-(q_2)^2h_{0,0,0,1}(\rho_2^{\mp +})^{q_2/2}
$$

$$
s_1\equiv \mathrm{sgn}[(|\delta|\alpha+|\eta|\gamma)^2-4|\delta\eta|\beta^2]. \tag{11}
$$

Then, one can prove that, according to the different values of these signs, one has two possible types of stability of the four families:

$$
\begin{aligned}
s_1>0 &\Rightarrow 2\mathrm{EH}+\mathrm{EE}+\mathrm{HH}, \\
s_1<0 &\Rightarrow 2\mathrm{CI}+2\mathrm{EH},
\end{aligned}
\tag{12}
$$

where EE is equal to the bi-elliptic fixed points, EH is equal to the elliptic-hyperbolic fixed point, HH is equal to the bi-hyperbolic fixed points, and CI is equal to the complex instability. A similar result holds for the other cases of double resonances. The analysis of the physical meaning of these two different stability situations needs a deeper investigation. It must be pointed out that these results are valid in the neighborhood of the fixed point located in the origin: one could have other combinations of stability when the resonance is sufficiently far from the origin.

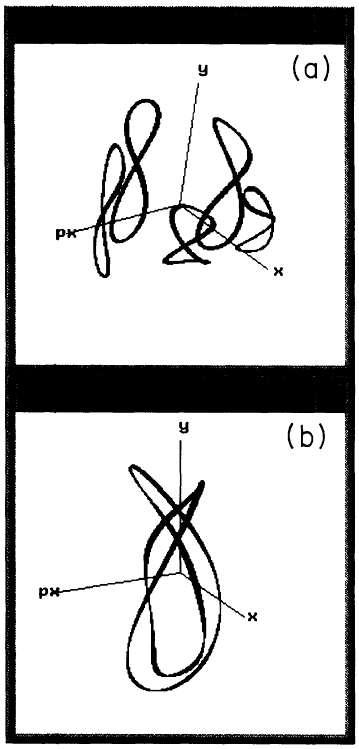

FIG. 1. Projection on a 3D space of the iterates of the 4D Hénon mapping: neighborhood of a single-resonance ($q=5$) elliptic line (a) and neighborhood of a coupled-resonance ($p=2$, $q=3$) elliptic line (b).

Numerical check

The numerical check of this conjecture has been performed for different models using a computer code for the visualization of projections of 4D orbits, a code for computing the interpolating Hamiltonian [13], and a code for computing the fixed points, based on a 4D generalization of the method given in Ref. [14]. Elliptic fixed lines relative to both uncoupled and coupled resonances have been found for different models. We have considered the 4D Hénon map:

$$z_1'=e^{i\omega_1}\left(z_1-\frac{i}{4}[(z_1+z_1^*)^2-(z_2+z_2^*)^2]\right)$$
$$z_2'=e^{i\omega_2}\left(z_2+\frac{i}{2}[(z_1+z_1^*)^2(z_2+z_2^*)]\right). \tag{13}$$

In Fig. 1(a) we display the projection on a 3D space (x,p_x,y) (where $z_1=x-ip_x$ and $z_2=y-ip_y$) of the stable neighborhood of a single-uncoupled resonance elliptic fixed line of period $q=5$. The linear frequencies were fixed at $\omega_1/(2\pi)=0.205$ and $\omega_2/(2\pi)=0.6180$; 50 000 iterates of a suitable initial condition are plotted. In Fig. 1(b) we display the 3D projection of the stable neighborhood of a single-coupled resonance elliptic fixed line, $p=2$ and $q=3$, for the same model with $\omega_1/(2\pi)=0.638$ and $\omega_2/(2\pi)=0.412$. One can see that the topology of the orbits is consistent with the above-quoted scheme. The case of the double resonance has also been analyzed for different models: we do not give the results here for the sake of brevity. A more detailed exposition of the analytical computations and of the numerical check will be described in a forthcoming paper.

ACKNOWLEDGMENTS

We would like to acknowledge Professor Vrahatis and Dr. Bazzani for important contributions. We also want to thank Professor Turchetti, Professor Ramis, Dr. F. Schmidt, and Dr. G. M. Todesco for useful and stimulating discussions.

[1] C. Froeschlé, Astron. Astrophys. **16**, 172 (1972).
[2] J. Moser, Lect. Notes in Math. **597**, 464 (1976).
[3] G. Contopoulos, Physica D **11**, 179 (1984).
[4] V. I. Arnold and A. Avez, *Ergodic Problems of Classical Mechanics* (Benjamin, New York, 1968).
[5] F. Schmidt, DESY HERA Report No. 88-02 (1988) (unpublished).
[6] A. D. Brjuno, Trans. Mosc. Math. Soc. **25**, 131 (1971).
[7] G. Servizi and G. Turchetti, Nuovo Cimento B **95**, 121 (1986).
[8] A. Bazzani, M. Giovannozzi, G. Servizi, E. Todesco, and G. Turchetti, Physica D **64**, 66 (1993).
[9] E. Forest, M. Berz, and J. Irwin, Part. Accel. **24**, 91 (1989).
[10] W. Scandale, F. Schmidt, and E. Todesco, Part. Accel. **35**, 53 (1991).
[11] F. Zimmermann, DESY Report No. 93-059 (1993) (unpublished).
[12] A. Bazzani, E. Todesco, G. Turchetti and G. Servizi, CERN Yellow Report No. 94-02, (1994) (unpublished).
[13] A. Bazzani, M. Giovannozzi, and E. Todesco, Comput. Phys. Commun. (to be published).
[14] M. N. Vrahatis, G. Servizi, G. Turchetti, and T. C. Bountis, CERN SL (AP) Report No. 93-06 (1993) (unpublished).

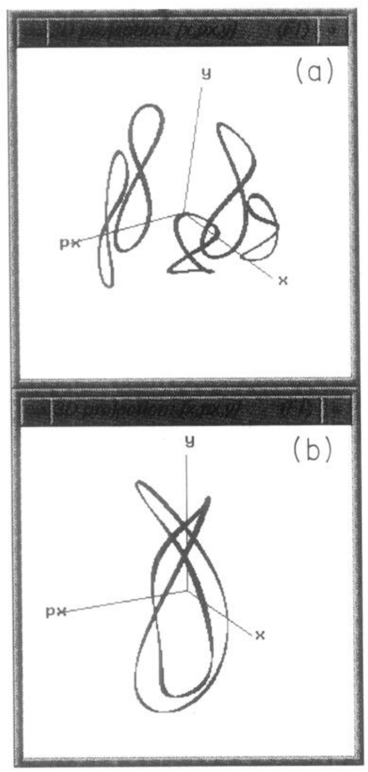

FIG. 1. Projection on a 3D space of the iterates of the 4D Hénon mapping: neighborhood of a single-resonance ($q=5$) elliptic line (a) and neighborhood of a coupled-resonance ($p=2,\ \ q=3$) elliptic line (b).